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Linear stability analysis of the Hele-Shaw cell with lifting plates

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Abstract. The first stages of finger formation in a Hele–Shaw cell with lifting plates are investigated by means of linear stability analysis. At the beginning of lifting the square of the wavenumber of the dominant mode results to be proportional to the lifting rate (in qualitative agreement with the available experimental data), to the square of the length of the cell occupied by the more viscous fluid, and inversely proportional to the cube of the cell gap. This dependence on the cell parameters is significantly different of that found in the standard cell. On the other hand, our results show that the wavelength of the dominant mode decreases with lifting time, also in agreement with several experimental observations.

PACS. 47.20.-k Hydrodynamics stability – 68.10.-m Fluid surfaces and fluid-fluid interfaces

1 Introduction

Despite of the great effort devoted lately to improve the understanding of Saffman-Taylor (ST) instabilities [1–4], many related experimental facts still lack a sound explanation [3]. Here we are interested in the experimental observations on Hele-Shaw (HS) cells with lifting plates [5]. This variation with respect to the standard constant gap HS cell was suggested as a way to bring the ST problem closer to directional solidification [6,7]. In this experiment (see Fig. 1), instead of applying pressure to the less viscous fluid, the upper plate is lifted at the less viscous side (commonly air) at a fixed rate. It seems clear that the lifting of the upper plate will promote a pressure gradient analogous to the temperature gradient present in directional solidification. An interesting variation of this experiment is the Hele–Shaw cell with a small gap gradient investigated by Zhao *et al.* [8] (see also Ref. [9]).

The main experimental results obtained by Ben–Jacob *et al.* [5] on a cell in which the bottom plate had a square lattice grooved on it are: i) at low lifting rates fingers are formed, whereas as the lifting velocity is increased, fingers turn into dendrites; and, ii) the spacing of the dendrites decreases as the lifting rate is increased, and does also depend on the initial plate spacing. On the other hand, in the more recent work of La Roche *et al.* [10] it was reported that, as lifting proceeds, the branches in the dendrites become thicker and the fractal dimension of the aggregate decreases. Although the experimental results for the HS with lifting plates have been analysed by means of a simplified version of the growth law [10,11], a detailed linear stability analysis of this system is still lacking.

In this work we present a study of growth instabilities in Hele–Shaw cells with lifting plates. We first derive the basic equation (growth law), which results to be different from that proposed by Ben–Jacob *et al.* [5]. In particular our equation is not homogeneous [10], and thus the comparison with the simpler version of the directional solidification problem is not so evident. Then we discuss the boundary conditions and carry out the linear stability analysis. Our results qualitatively explain the experimental data outlined in the preceeding paragraph.

The paper will be organized in five sections. In Section 2 we will present the continuum equations used, whose perturbative solution is studied in detail in Section 3. The physical consequences and implications of these solutions is the subject for Section 4. Finally, in Section 5, we will summarize the results.

2 Basic equations and boundary conditions

Flow in the Hele-Shaw cell is governed by the Navier-Stokes equation [12,13]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\mathbf{v}\,, \tag{1}$$

where $\mathbf{v}$ is the speed of the fluid and P the pressure. ρ and μ are the density and the viscosity of the fluid, respectively. For small Reynolds numbers and assuming that the time derivative of the fluid velocity is much smaller than its spatial derivatives, this equation reduces to

$$\nabla P = \mu\nabla^2\mathbf{v}. \tag{2}$$

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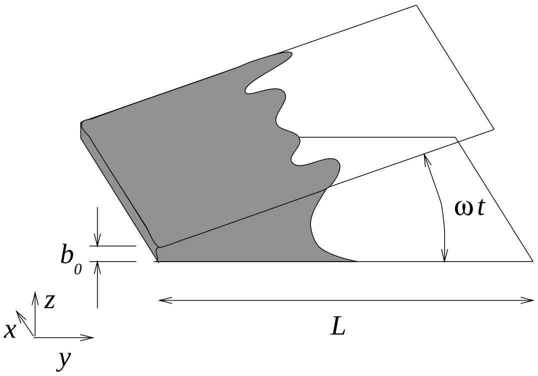

Fig. 1. Schematic representation of the Hele-Shaw cell with lifting plates, showing the parameters used in the text.

Averaging over the the cell gap leads to the Poiseuille-Darcy equation for the mean velocity of the fluid,

$$\mathbf{v}_j = -\frac{b^2}{12\mu_j}\nabla P_j, \tag{3}$$

where $\mathbf{v}_j$, μ_j, and P_j are the velocity, viscosity and pressure field of fluid j ($j = 1,\ 2$); and b is the gap of the cell. In order to obtain an equation for the pressure field we need to combine equation (3) with mass conservation. In the present case the latter deserves a careful consideration. Let the HS cell lie in the $x-y$ plane, the y–axis being the direction of motion of the fluids, and the origin of coordinates be at the closed (fixed) end of the cell (see Fig. 1). The gap of the cell varies as

$$b(y,t) = b_0 + y\,\tan(\omega t)\ , \tag{4}$$

where ω is the lifting angular speed (we shall hereafter call $a = \tan(\omega t)$). As a consequence, the mass within a thin column of height b changes as $\delta m/\delta t \propto \delta b/\delta t$ (where the density of the fluid ρ is assumed to be constant). Then, the equation which describes mass conservation reads

$$\frac{\partial b}{\partial t} = -\nabla\cdot(b\mathbf{v}_j)\ . \tag{5}$$

It should be here noted that mass (and density) conservation requires that, neither bubbles are formed nor drops of the displaced fluid are left behind during the lifting process. This condition, although unlikely accomplished in actual experiments, is probably unavoidable in analytical calculations. Equation (5), combined with the Poiseuille-Darcy equation (Eq. (3)), gives the differential equation (growth law) which governs flow in the HS cell with lifting plates,

$$\nabla^2 P_j + \frac{3a}{b}\frac{\partial P_j}{\partial y} = \frac{12\mu_j \dot{a} y}{b^3}\ , \tag{6}$$

where $\dot{a}$ expresses the derivative of a with respect to time. It is interesting to note that this is an inhomogeneous equation as opposed to the homogeneous one reported in reference [5]. The inhomogeneous term comes from the time dependence of the gap (Eq. (4)) and it is not present in the growth law for the HS with a gap gradient, as already found by Zhao *et al.* [3,8]. Note, however, that the inhomogeneous term is essential in this case, as no fluid is injected at the open end of the cell to promote fluid motion (see below). We also note that including the spatial dependence of the gap b in equation (5) leads to a factor 3 in the second term of equation (6), as opposed to the factor 2 reported in reference [5] and in agreement with Zhao *et al.* [8]. We should remark that neglecting the second term in the l.h.s. of equation (6) — as done in reference [10] — does not qualitatively change the results, in line with Zhao *et al.* [8]. The latter authors reported that, for most experimental conditions, a linear stability analysis of the HS cell with a gap gradient gaves similar results than for the standard HS cell. As regards the comparison with the directional solidification problem, we note that equation (6) is similar to that which describes directional solidification in some *unsteady*-state [7].

In discussing the boundary conditions let L be the length of the cell and L_i the length of the zone occupied by the more viscous fluid at $t = 0$. On the other hand, and in order to carry out the linear stability analysis, we assume that the interface between the two fluids, instead of being flat, is slightly perturbed as $y_\mathrm{p} = y_i + \delta \mathrm{e}^{iqx}$, where $\delta = \delta_0 \mathrm{e}^{\gamma t}$ is the time–dependent amplitude of the perturbation (assumed to be much smaller than y_i) and q its wavenumber.

The first boundary condition accounts for the fact that the fixed end of the cell ($y = 0$) is closed,

$$v_{1y}(0) = 0. \tag{7}$$

On the other hand, the velocities of the two fluids must be equal at the interface

$$v_{1y}(y_\mathrm{p}) = v_{2y}(y_\mathrm{p}). \tag{8}$$

Finally the forces which the fluids exert on each other at the common interface must be equal and opposite. This condition can be approximated by

$$P_1(y_\mathrm{p}) - P_2(y_\mathrm{p}) = \sigma\left[\frac{1}{R_x(y_\mathrm{p})} + \frac{1}{R_z(y_\mathrm{p})}\right], \tag{9}$$

where σ is the surface-tension (the interfacial tension associated to the interface between the two fluids) and R_x, R_z, the principle radii of curvature of the interface at a given point,

$$\frac{1}{R_x(y_\mathrm{p})} = -\frac{\partial^2 y_\mathrm{p}}{\partial x^2}, \tag{10a}$$

$$\frac{1}{R_z(y_\mathrm{p})} \approx \frac{1}{b(y_i,t)}\ . \tag{10b}$$

The latter equation is related to wetting effects. Although, as already reported by other authors [8,13,14], we have found that these effects give a negligible contribution, we have kept this term all throughout the calculation.

3 Linear stability analysis

Once the plane interface is perturbed, the most general solution of equation (6) can be written as

$$P_j(x,y) = f_j(y) + g_j(y)\mathrm{e}^{iqx} \ , \qquad (11)$$

where $g_j(y)$ is proportional to the amplitude of the perturbation δ. Introducing equation (11) into the growth law (Eq. (6)) we obtain

$$\frac{\partial^2 f_j}{\partial y^2} + \frac{3a}{b}\frac{\partial f_j}{\partial y} = \frac{12\mu_j \dot{a} y}{b^3}, \qquad (12a)$$

$$\frac{\partial^2 g_j}{\partial y^2} + \frac{3a}{b}\frac{\partial g_j}{\partial y} - q^2 g_j = 0. \qquad (12b)$$

The boundary conditions for the fluid velocity at the closed end give

$$\left.\frac{\partial f_1}{\partial y}\right|_{y=0} = 0, \qquad (13a)$$

$$\left.\frac{\partial g_1}{\partial y}\right|_{y=0} = 0. \qquad (13b)$$

Note that in order to obtain physically meaningful solutions we should also require that the perturbation be damped in the open end of the cell, this means that $(\partial g_2/\partial y)|_{y=L} = 0$. This assumption will be valid whenever the viscosity μ_2 is very small (note that, in the present work, we will analyze experiments in which fluid 2 is air). The continuity of the velocity at the interface between the two fluids leads to (up to first order in δ):

$$\frac{1}{\mu_1}\left.\frac{\partial f_1}{\partial y}\right|_{y=y_i} = \frac{1}{\mu_2}\left.\frac{\partial f_2}{\partial y}\right|_{y=y_i}, \qquad (14a)$$

$$\frac{1}{\mu_1}\left.\left(\delta\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial g_1}{\partial y}\right)\right|_{y=y_i} = \frac{1}{\mu_2}\left.\left(\delta\frac{\partial^2 f_2}{\partial y^2} + \frac{\partial g_2}{\partial y}\right)\right|_{y=y_i}. \qquad (14b)$$

Finally, the continuity of forces at the interface gives

$$\left[f_1 - f_2 - \frac{\sigma}{R_z}\right]_{y=y_i} = 0, \qquad (15a)$$

$$[g_1 - g_2(y)]_{y=y_i} = -\left[\frac{\partial(f_1 - f_2)}{\partial y}\right]_{y=y_i} + \sigma q^2. \qquad (15b)$$

The solution of equation (12a), using the boundary conditions (13a, 14a), can be written as

$$f_j = \frac{12\mu_j \dot{a}}{a^3}\left(\frac{b_0}{b(y,t)} + \frac{1}{2}\ln b(y,t) - \frac{b_0^2}{4b(y,t)^2}\right) + A_j, \quad (16)$$

where A_j ($j = 1, 2$) are constants that can be determined from the boundary condition of equation (15a) (we will not need them here). It should be noted that, had equation (6) been homogeneous, the solution of equation (12a) would have been, $f_j = A_j - B_j/b(y,t)^2$, which, after proper use of the boundary conditions, gives a uniform pressure field and, therefore, no fluid motion, as remarked above. On the other hand the solution of equation (12b) is

$$g_j = \frac{1}{\xi}\left[C_j I_1(\xi) + D_j K_1(\xi)\right] \ , \qquad j = 1,2, \qquad (17)$$

and its derivative,

$$\frac{\partial g_j}{\partial y} = \frac{q}{\xi}\left[C_j I_2(\xi) - D_j K_2(\xi)\right] \ , \qquad j = 1,2, \qquad (18)$$

where $\xi = qb(y,t)/a$, and $I_1(\xi)$, $I_2(\xi)$, and $K_1(\xi)$, $K_2(\xi)$ are the first and the second order modified Bessel functions. The constants C_j, D_j have to be determined from the boundary conditions,

$$C_1 I_2(\xi_0) - D_1 K_2(\xi_0) = 0, \qquad (19a)$$

$$C_2 I_2(\xi_L) - D_2 K_2(\xi_L) = 0, \qquad (19b)$$

$$\left(\frac{C_1}{\mu_1} - \frac{C_2}{\mu_2}\right) I_2(\xi_i) - \left(\frac{D_1}{\mu_1} - \frac{D_2}{\mu_2}\right) K_2(\xi_i) = 0, \qquad (19c)$$

$$(C_1 - C_2) I_1(\xi_i) + (D_1 - D_2) K_1(\xi_i) = \epsilon\delta, \qquad (19d)$$

where $\xi_0 = qb_0/a$, $\xi_i = q(b_0 + ay_i)/a$, $\xi_L = q(b_0 + aL)/a$, and ϵ is given by

$$\epsilon = \xi_i\left[\frac{12(\mu_1 - \mu_2)V_i(y_i)}{b(y_i,t)^2} + \sigma q^2\right], \qquad (20)$$

where the velocity of the interface between the two fluids is given by

$$V_i(t) = -\frac{b^2}{12\mu_j}\left.\frac{\partial f_j}{\partial y}\right|_{y=y_i} = \dot{y}_i = -\frac{\dot{a} y_i^2}{2b(y_i,t)}, \qquad (21)$$

independent of j. Note that, due to the choice of the origin of coordinates, this velocity is negative. Integrating this velocity gives an expression for the position of the interface:

$$y_i(t) = \frac{\sqrt{b_0^2 + 2L_i b_0 a} - b_0}{a}. \qquad (22)$$

Note that $y_i(0) = L_i$. This result coincides with that obtained from mass conservation, $\frac{1}{2}ay_i^2 + b_0 y_i = b_0 L$. We note that neglecting the second term in the l.h.s. of the growth equation (Eq. (6)) is inconsistent with mass conservation.

Now, we have all the ingredients to calculate the instantaneous velocity of the interface (growth rate), which can be calculated from the Poiseuille-Darcy equation. The result is

$$\begin{aligned} v_{1y}(y_\mathrm{p}) &= \dot{y_\mathrm{p}} = \dot{y_i} + \dot{\delta}\mathrm{e}^{iqx} \\ &= -\frac{b^2(y_\mathrm{p},t)}{12\mu_1}\left.\frac{\partial(f_1 + g_1\mathrm{e}^{iqx})}{\partial y}\right|_{y=y_\mathrm{p}}. \end{aligned} \qquad (23)$$

From which we obtain the velocity of the perturbation γ

$$\gamma = \frac{\dot{\delta}}{\delta} = -\frac{\dot{b}(y_i,t)}{b(y_i,t)} - \frac{ab(y_i,t)\epsilon}{12} Z(\xi_i,\xi_L), \qquad (24)$$

where,

$$Z(\xi_i,\xi_r) = \frac{I_2(\xi_i)}{I_1(\xi_i)} \times \left(\mu_1 \frac{[G_2(\xi_0)+G_1(\xi_i)]}{[G_2(\xi_0)-G_2(\xi_i)]} - \mu_2 \frac{[G_2(\xi_L)+G_1(\xi_i)]}{[G_2(\xi_L)-G_2(\xi_i)]}\right)^{-1}, \qquad (25)$$

and,

$$G_j(\xi) = K_j(\xi)/I_j(\xi)\ , \qquad j = 1, 2\ . \qquad (26)$$

At the very beginning of liftting $1 \ll \xi_0 \ll \xi_i \ll \xi_L$. As a consequence, $G_1(\xi_i) \simeq G_2(\xi_i)$. On the other hand, and due to the exponential behavior of the Bessel and Hankel functions, $G_2(\xi_0) \gg G_2(\xi_i) \gg G_2(\xi_L)$. Thus, equation (24) can be approximated as

$$\gamma = -\frac{\dot{b}(y_i,t)}{b(y_i,t)} - \frac{b^2(y_i,t)}{\mu_1+\mu_2} q \left[\frac{12(\mu_1-\mu_2)V_i(t)}{b^2(y_i,t)} + \sigma q^2\right]. \qquad (27)$$

The wavenumber of the dominant mode q_{d} results to be

$$q_{\mathrm{d}}^2 = -\frac{4(\mu_1-\mu_2)V_i(t)}{\sigma b^2(y_i,t)}\ \cdot \qquad (28)$$

This result is quite similar to that obtained for the standard Hele-Shaw cell. The only difference resides upon the fact that the interface velocity $V_i(t)$ and the gap $b(y_i,t)$ now depend on time. As regards the cutoff wavenumber, we note that if the first term in the r.h.s of equation (27) is neglected (in fact it is quite small for most of the experimental configurations and conditions), the result is again equivalent to that of the standard cell ($q_{\mathrm{c}}^2 = 3q_{\mathrm{d}}^2$). If it is not neglected, a minimum wavenumber, below which the system is stable, is also found.

At $t = 0$, the dominant mode is

$$q_{\mathrm{d}}^2 = -\frac{4(\mu_1-\mu_2)}{\sigma b_0^2} V_i(0), \qquad (29)$$

where $V_i(0)$ is the velocity of the interface at $t = 0$,

$$V_i(0) = -\frac{\omega L_i^2}{2b_0}\ \cdot \qquad (30)$$

$V_i(0)$ introduces a dependence on the cell parameters of the wavenumber of the dominant mode, not present in the standard cell. In particular, q_{d} depends on the square of the length occupied by the displaced (more viscous) fluid. It is also inversely proportional to the gap, leading to a b_0^{-3} dependence of q_{d}, as opposed to b_0^{-2} in the standard cell. These differences could be easily checked experimentally.

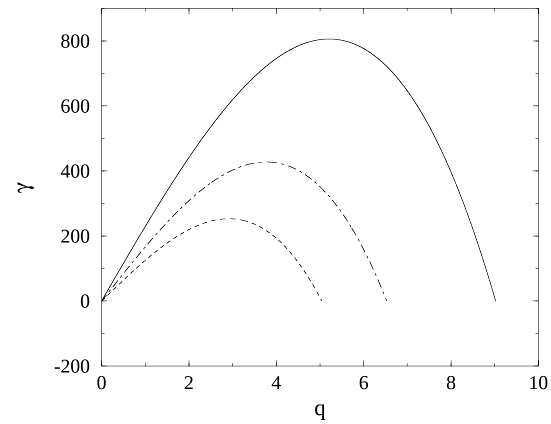

Fig. 2. Results for the velocity of the perturbation γ (rad/s) as a function of its wavenumber q (cm^{-1}), for several times after lifting is initiated: $t = 0$ s (continuous line), $t = 0.5$ s (chain line) and $t = 1$ s (dashed line). The results correspond to the cell parameters of reference [7], namely: air/glycerine ($\mu = 65$ mPa and $\sigma = 29.5$ N/m) in a cell with a gap of 2.5 mm and a length of 1 m (we assume that glycerine fills the whole cell). The lifting rate is $\omega = 0.001$ rad/s.

4 Numerical results and discussion

At the beginning of lifting ($t = 0$) the process is governed by equation (30). In actual experiments, as the less viscous fluid is commonly air, $\mu_2 \approx 0$, and the wavelength of the dominant and the cutoff modes are given by expressions identical to those for the standard cell, namely, $\lambda_{\mathrm{d}}/b_0 = \pi(\sigma/\mu_1 V_i(0))^{1/2}$, and $\lambda_{\mathrm{c}} = \lambda_{\mathrm{d}}/\sqrt{3}$, respectively. In order to estimate the wavelength of the dominant mode we consider the experimental set up investigated by Zhao *et al.* [8]. These authors used air/glycerine ($\mu = 65$ mPa and $\sigma = 29.5$ N/m) in a cell with a gap of 2.5 mm and a length of 1 m (we assume that glycerine fills the whole cell). Taking $\omega = 0.001$ rad/s, $\lambda_{\mathrm{d}} \approx 1.2$ cm. This result decreases in an order of magnitude if, as done by Ben–Jacob *et al.* [5], the lifting rate is increased in two orders of magnitude. Although we cannot precisely compare our results with the data obtained by the latter authors as they do not give important parameters such as the length and gap of the cell, the decrease in the spacing of the dendrites as the lifting rate is increased reported by Ben–Jacob *et al.* [5], is clearly reproduced by our analysis.

If instead of glycerine we consider water, at $t = 0$ and for the same cell parameters, the wavelength of the dominant mode results to be ≈ 0.4 cm^{-1}, which leads to a wavelength of ≈ 16 cm. This wavelength is of the order of the full width of standard cells and therefore one should not expect the formation of fingers. Further we note that the velocity of the perturbation results to be at least an order of magnitude smaller than for glycerine, reinforcing our view that the flat front would be rather stable. Note that this result is also obtained for the standard cell and that it is in agreement with the experimental observations. The only way to increase the tendency towards instabilities would be to increase the velocity of the interface which

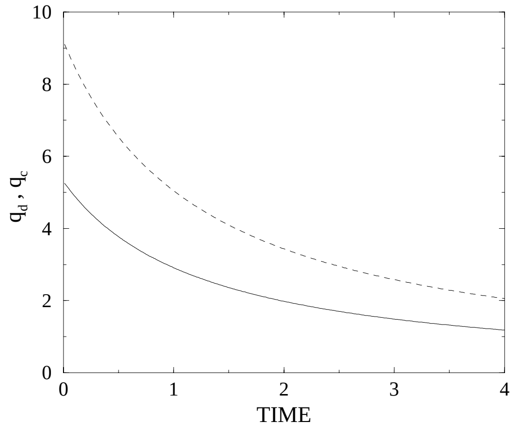

Fig. 3. Wavenumbers (cm^{-1}) of the dominant (q_d, continuous line) and cutoff (q_c, broken line) modes as a function of time (s) in the Hele–Shaw with lifting plates investigated in this work. The results correspond to the parameters given in the caption of Figure 1.

in the present case can be accomplished by increasing L and the lifting rate and/or decreasing b_0.

As lifting proceeds, fingers begin to form and a linear stability analysis, such as that carried out here, is no longer strictly valid. However, as the overall shape of the front is flat, we expect the linear stability analysis to give useful hints even beyond that point. In Figure 2 we report our results for the velocity at which the perturbation propagates as a function of the perturbation wavenumber, for several times after lifting was initiated. In the calculations the first term in the r.h.s. of equation (27) was neglected. The results correspond to the cell parameters given in the preceeding paragraph and a lifting rate of $\omega = 0.001$ rad/s. It is noted that the velocity of the perturbation γ strongly decreases with time. In fact the value at its maximum is reduced in more than a factor of 5 after lifting the cell for 1 second. The wavenumber at which γ shows a maximum (dominant mode) and the cutoff wavenumber do also decrease with time (see Fig. 3). The decrease of the velocity of propagation of the perturbation suggest a lower tendency towards instabilities and, thus, a smaller fractal dimension of the aggregates (this is compatible with a decreasing q_d). These results agree with the experimental observations (and numerical simulations) reported by La Roche *et al.* [10] which indicate that, as lifting proceeds, the branches of the dendrites become thicker and the fractal dimension of the aggregate decreases. It is also interesting to note that the decrease in the strength of the instability with the time of lifting indicates that the front will remain flat on average, instead of evolving into a single finger as in the standard HS cell [3].

5 Conclusions

Summarizing, we have presented a linear stability study of the HS with lifting plates. We have first derived the basic equations which result to be that of the directional solidification problem under some unsteady conditions. Despite the simplifications made to do the analytical work (we do not treat completely the wetting effects and neglect the fluid that stays attached to the plates), the results for the wavelength of the dominant mode λ_c seem to be compatible with the available experimental data. In particular we obtain that λ_c decreases with the lifting velocity and the lifting time, in agreement with several experimental observations reported in references [5, 10]. Nonetheless, more experimental studies of the VHSC are required in order to allow a full (quantitative) test of the linear stability analysis discussed in the present work.

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